1. **See Excel Answer Sheet**
2. **See Excel Answer Sheet**

**3. Smallpox vaccination**

The origin of smallpox is uncertain, but it is believed to have originated in Africa and then spread to India and China thousands of years ago. The first recorded smallpox epidemic was in 1350 BC during the Egyptian-Hittite war. Smallpox reached Europe between the 5th and 7th centuries and was present in major European cities by the 18th century. Epidemics occurred in the North American colonies in the 17th and 18th centuries. At one time smallpox was a significant disease in every country throughout the world except Australia and a few isolated islands. Millions of people died in Europe and Mexico as a result of widespread smallpox epidemics.

The fall of smallpox began with the realization that survivors of the disease were immune for the rest of their lives. This led to the practice of variolation - a process of exposing a healthy person to infected material from a person with smallpox in the hopes of producing a mild disease that provided immunity from further infection. The first written account of variolation describes a Buddhist nun practicing around 1022 to 1063 AD. She would grind up scabs taken from a person infected with smallpox into a powder, and then blow it into the nostrils of a non-immune person. By the 1700's, this method of variolation was common practice in China, India, and Turkey. In the late 1700's European physicians used this and other methods of variolation, but reported "devastating" results in some cases. Overall, 2% to 3% of people who were variolated died of smallpox.

The next step towards the eradication of smallpox occurred with the observation by English physician, Edward Jenner, that milkmaids who developed cowpox, a less serious disease, did not develop the deadly smallpox. In 1796, Jenner took the fluid from a cowpox pustule on a dairymaid's hand and inoculated an 8-year-old boy. Six weeks later, he exposed the boy to smallpox, and the boy did not develop any symptoms. Jenner coined the term "vaccine" from the word "vaca" which means "cow" in Latin.

If the chance of dying from being vaccinated from the first primitive vaccines were 1 in 200 while the chance of dying from smallpox if you contracted the disease were 1 in 7, should you be vaccinated? Discuss

**To see whether you should get vaccinated, you need to know something about the chance of contracting smallpox without being vaccinated. If the chance is, say 1/10, then the chance you would die from smallpox without a vaccination is the product of two numbers: 1/10 and 1/7 = 1/70. This is still larger than the chance of dying if you are vaccinated, 1/200. So, presuming that the vaccination (if you live through it) is 100% effective at protecting you from the disease, you should get vaccinated. On the other hand, if the chance of contracting smallpox is even smaller, say 1 out of 100, then the chance of dying without the vaccine would be the product of 1/7 and 1/100 = 1/700. In this case, vaccination would not be a good deal.**

1. **Absent Minded Professor**

An absent minded professor has 3 sock drawers. He knows he has 2 blue socks in one drawer, 2 red socks in a second drawer, and the third drawer contains one blue and one red sock. The problem is, he doesn’t remember which drawer is which. To solve his conundrum, he picks a drawer at random, closes his eyes, reaches in and draws out a sock at random. Afterwards, he observes that it is a red sock. What is the probability that the single remaining sock in that drawer is red?

**Since the drawer is chosen at random, there is, if you were given no further information, a 1/3 chance that it could be any of the three. Note, however, that we are given additional information: the man draws a random sock from the drawer which happens to be red. Since it is impossible to draw a red sock from a drawer containing only blue socks, the drawer containing two blue socks is no longer a possibility. The two remaining choices are: the drawer with two red socks and the drawer with one red and one blue sock. Note that only in the former case will the remaining sock in the drawer be red. It might seem that, with only two drawers remaining, the odds are 50-50. But this would be wrong as the following calculation shows:**

**Scenario analysis: Imagine 300,000 people performing this experiment. We would expect 100,000 of them to choose the RR drawer, 100,000 to choose the RB drawer, and 100,000 of them to choose the BB drawer. Of the ones who chose the RR drawer, all 100,000 would draw out a red sock, but only 50,000 of those who chose the RB drawer would do so. Thus, on average, 150,000 people will have drawn out a red sock. If we look only at those who have drawn out a red sock, 100,000 of them will have done so from the RR drawer and 50,000 of them will have done so from the RB drawer. So the chances are 100,000 out of 150,000 (or, equivalently, 2 out of 3) that someone who drew out a red sock will have done so from the RR drawer. Thus, the probability that the remaining sock in the drawer is red, given that the first one drawn out was red is 2/3.**

**Following is the same problem worked out using the formulaic notation from week 2 slides:**

**Let RR denote the event that he chose drawer with 2 red socks.**

**Let RB denote the event that he chose the drawer with one red and one blue sock.**

**Let BB denote the event that he chose the drawer with two blue socks.**

**P{RR} = P{RB} = P{BB} = 1/3**

**Let R denote the event that the first sock withdrawn is red and B the event that the first sock withdrawn is blue. Note that (from the law of total probability)**

**P{R} = P{R ∩ RR} + P{R ∩ RB} + P{R ∩ BB}**

**= P{R | RR}P{RR} + P{R |RB}P{RB} + P{R | BB}P{BB}**

**= 1 x 1/3 + ½ x 1/3 + 0 x 1/3 = 1/3 + 1/6 = ½**

**An exactly analogous calculation shows that P{B} = ½ also.**

**We seek the probability of RR given R. That is, P{RR | R}.**

**From the law of total probability, we know that:**

1. **P{RR} = P{RR ∩ R} + P{RR ∩ B} = P{RR | R} x P{R} + P{RR | B} x P{B}**

**We already know that:**

**P{RR} = 1/3, P{RR | B} = 0, P{R} = ½**

**Since P{RR | B} = 0, equation (1) reduces to:**

**P{RR} = P{RR | R} x P{R}**

**Since we know P{RR} and P{R} , this reduces to:**

**1/3 = P{RR | R} x ½ and we can solve for P{RR | R} = 2/3.**

**The reason it works out this way is that the man has only a 50% chance of drawing a red sock from the RB drawer on his first try, while it is a 100% chance if he has the RR drawer. Another way to think about the problem is to imagine the following modification: 1 drawer contains 100 red socks, a second drawer contains 100 blue socks, and the third drawer contains 99 blue socks and 1 red sock. In this case, if the man drew a red sock from his chosen drawer on the first try, would you bet against him having the drawer with 100 red socks?**

1. **An Urn Problem**

Suppose you have two urns, each containing 10 balls. One urn has 7 blue and 3 red balls, and the second urn has 3 blue and 7 red balls. The urns are opaque and their outside appearance is identical, so we can’t tell which urn is which. Suppose you select an urn at random. You then choose a random ball from the urn and observe it is red. You then replace it, select a random ball from the urn again and observe it is red. Again you replace and select a third ball at random, and it is red. You replace a final time, select a fourth ball at random, and it is red. What is the probability you chose the red-heavy urn? Hint: imagine having 20,000 people performing the same experiment, randomly choosing 1 of the two urns and randomly drawing out 1 of the 10 balls (with replacement) four different times.

Let R denote the event that you choose the red heavy urn and B the event that you choose the blue heavy urn. Let RRRR denote the event that you draw 4 reds with replacement.

We seek P{R | RRRR}.

**Scenario Analysis in Red: Of the 20,000 people, we would expect half to choose the red heavy urn and half to choose the blue heavy urn.**

**P{R} = P{B} = 1/2**

**If someone chose the red heavy urn, the probability they will draw a red ball on a single try is 7/10 every try. The probability that they will draw red balls four times is, therefore, 7/10 x 7/10 x 7/10 x 7/10 = 0.2401. Since there are 10,000 of them, we would expect 2401 to draw 4 reds.**

**P{RRRR | R} = 0.2401**

**For someone choosing the blue heavy urn, the probability of drawing 4 reds is given by the product: 3/10 x 3/10 x 3/10 x 3/10 = 0.0081. Since there are 10,000 of them, we would expect 81 to draw 4 reds.**

**P{RRRR | B} = 0.0081**

**Out of our total population, we would expect 2401 + 81 = 2482 (out of 20,000) to draw 4 reds.**

**P{RRRR} = P{RRRR ∩ R} + P{RRRR ∩ B}**

**= P{RRRR |R}P{R} + P{RRRR | B}P{B}**

**= 0.2401 \* ½ + 0.0081 \* ½ = 0.1241**

**Note also that**

**P{RRRR ∩ R} = 0.2401 x ½ = 0.12005**

**P{RRRR ∩ B} = 0.0081 x ½ = 0.00405**

**Of these, 2401(out of 20,000) would be drawing from the red heavy urn and 81 (out of 20,0000) would be drawing from the blue-heavy urn. Thus, the probability of having a red heavy urn given that you draw four reds is 2401/2482 = 96.7% (approximately).**

**P{R | RRRR} = P{RRRR ∩ R}/P{RRRR} = 0.12005/0.1241 = 96.7%**